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The Picard Group of a Monoid Domain

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INTRODUCTION

Let R be an integral domain and S a nonzero commutative cancellative torsion-free monoid; thus the monoid ring $R[S]$ is also an integral domain. In this paper we determine necessary and sufficient conditions for the natural homomorphism $\mathrm{Pic}(R) \to \mathrm{Pic}(R[S])$ to be an isomorphism. Specifically, $\mathrm{Pic}(R) = \mathrm{Pic}(R[S])$ if and only if $R[S]$ is seminormal (equivalently, R and S are each seminormal) and $\mathrm{Pic}(R) = \mathrm{Pic}(R[H])$, where H is the group of invertible elements of S. Thus the problem of determining when $\mathrm{Pic}(R) = \mathrm{Pic}(R[S])$ for S a monoid is reduced to the problem of determining when $\mathrm{Pic}(R) = \mathrm{Pic}(R[H])$ for H a group. Our theorem has several particularly interesting corollaries. First, if R is an integrally closed domain, then $\mathrm{Pic}(R) = \mathrm{Pic}(R[S])$ if and only if $R[S]$ is seminormal. Second, since a subring A of $R[\{X_\alpha, X_\alpha^{-1}\}]$ generated by monomials over R may be viewed as the monoid ring $R[S]$, where S is the submonoid of $\bigoplus \mathbb{Z} e_\alpha$ determined by the exponents of the monomials in A, a subring A of $K[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}]$ generated by monomials over the field K has trivial Picard group if and only if A is seminormal.

Our theorem generalizes several special known cases. The classic situation is when $S = \mathbb{Z}_+^n$. Then $\mathrm{Pic}(R) = \mathrm{Pic}(R[\mathbb{Z}_+^n])$ if and only if R is seminormal [9]. More generally, if S is any nonzero seminormal submonoid of $\mathbb{Z}_+^n$, then $\mathrm{Pic}(R) = \mathrm{Pic}(R[S])$ if and only if R is seminormal by [4, Theorem 1] because $R[S]$ is a $\mathbb{Z}_+$-graded ring with $R[S]_0 = R$. Thus by taking direct limits, $\mathrm{Pic}(R) = \mathrm{Pic}(R[S])$ if $R[S]$ is seminormal and $H = \langle 0 \rangle$. Finally, if $R[S]$ is integrally closed, then $\mathrm{Pic}(R) = \mathrm{Pic}(R[S])$ [1, Corollary 5.6].

One of the main reasons for the increased difficulty in the general case is that H need not be a direct summand of S. For example, while $\mathbb{Z}_+$-graded ring techniques are sufficient to show that $\mathrm{Pic}(R) = \mathrm{Pic}(R[S])$ when $R[S]$ is seminormal and $S \subset \mathbb{Z}_+^n$, more complicated techniques seem to be needed

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for $S \subset \mathbb{Z}^n$. Our proof is based on a careful analysis of each ring in a cartesian square and the corresponding Mayer–Vietoris sequence for (U, Pic). Such an analysis is greatly simplified (and clarified) by the extra generality of monoid rings rather than just considering subrings generated by monomials.

Section 1 contains the necessary background information on monoids and monoid rings and several technical lemmas. Our main theorem and several corollaries comprise Section 2. We close with some remarks and a question about finitely generated projective $K[S]$-modules.

1. Preliminaries

In this section we establish our notation, include some definitions and facts about integral domains and monoids, and prove several lemmas to be used in the proof of our main theorem. Throughout, R denotes an integral domain with quotient field K. The Picard group of R, $\mathrm{Pic}(R)$, is the group of isomorphism classes of finitely generated projective R-modules of rank one under $\otimes_R$, or equivalently, since here R is an integral domain, the group of invertible fractional ideals of R modulo the subgroup of principal fractional ideals. Recall that R is *seminormal* if whenever $r^2, r^3 \in R$ for some $r \in K$, then $r \in R$ (or equivalently, if $a^2 = b^3$ for $a, b \in R$, then $a = c^3$ and $b = c^2$ for some $c \in R$ (cf. [16])). Then $\mathrm{Pic}(R) = \mathrm{Pic}(R[X_1, ..., X_n])$ for all $n \geqslant 1$ if and only if R is seminormal [9]. A subring R of a ring A is a *retract* of A if there is a ring homomorphism $A \to R$ which is the identity on R. Since Pic is a functor, this induces a split monomorphism $\mathrm{Pic}(R) \to \mathrm{Pic}(A)$.

Throughout, S denotes a torsionless grading monoid. That is, S is a commutative cancellative monoid, written additively, and the quotient group $\langle S \rangle$ generated by S is a torsion-free abelian group. The monoid ring $R[S]$ is $R[\{X^s | s \in S\}]$ with $X^s X^t = X^{s+t}$ for each $s, t \in S$. We note that $R[S]$ is an integral domain if and only if R is an integral domain and S is a torsionless grading monoid [8, Theorem 8.1].

A particularly important case is when $\langle S \rangle = \bigoplus \mathbb{Z} e_\alpha$ is a free abelian group. Then the monoid ring $R[S]$ may be considered as a subring of $R[\{X_\alpha, X_\alpha^{-1}\}]$ generated by monomials over R; for example, $R[X_1, ..., X_n] = R[\mathbb{Z}_+^n]$ and $R[X_1, X_1^{-1}, ..., X_n, X_n^{-1}] = R[\mathbb{Z}^n]$. Also, $R[S]$ becomes an S-graded domain with $\deg rX^s = s$ for each $s \in S$ and $0 \neq r \in R$. A basic reference for semigroups and semigroup rings is [8].

We next review a few definitions and facts about semigroups. A nonempty subset J of S is an *ideal* of S if $J + S \subset J$. A proper ideal J of S is a *prime ideal* if $s + t \in J$ for $s, t \in S$ implies $s \in J$ or $t \in J$. Thus J is prime if and only if $S - J = \{s \in S | s \notin J\}$ is a submonoid of S. A proper ideal J of S is a *radical ideal* if $ns \in J$ for $n \geqslant 1$ and $s \in S$ implies $s \in J$. Given a semigroup ideal J of S, then $R[J] = \{\sum r_s X^s | r_s \in R, s \in J\}$ is an (S-homogeneous) ideal of $R[S]$. Note that $R[J]$ is a prime (resp., radical) ideal of $R[S]$ if and only if J is a prime (resp., radical) ideal of S. If J is a prime ideal of S, then the inclusion $R[S-J] \to R[S]$ induces an isomorphism of $R[S-J]$ onto $R[S]/R[J]$. So $R[S]/R[J] = R[S-J]$ is again a monoid domain and is a retract of $R[S]$ (cf. [8, Theorem 12.1]). If J is only assumed to be a radical ideal of S, then while $R[S]/R[J]$ need not be a monoid domain, each of its elements may be represented uniquely as $\sum r_s X^s$ with $r_s \in R$ and $s \in S - J$ and with multiplication defined by $X^s X^t = X^{s+t}$ if $s + t \in S - J$ and $X^s X^t = 0$ otherwise. Note that R is a retract of both $R[S]$ and $R[S]/R[J]$.

The group of units of a ring R will be denoted by $U(R)$. The set of invertible elements of a semigroup S also forms a group, namely $H = S \cap -S$. The monoid domain $R[S]$ has only trivial units, i.e., $U(R[S]) = \{rX^s | r \in U(R) \text{ and } s \in H\}$ [8, Theorem 11.1]. If J is a prime ideal of S, then $R[S]/R[J] = R[S-J]$ and $R[S]$ have the same units since S and $S-J$ have the same group of invertible elements. A more interesting result is

LEMMA 1. *Let R be an integral domain, S a torsionless grading monoid with group of invertible elements H, J a radical ideal of S, and $A = R[S]/R[J]$. Then $U(A) = \{rX^s | r \in U(R)$ and $s \in H\}$.*

Proof. Clearly each such rX^s is a unit in A since $H \subset S - J$. Suppose that $u = \sum r_s X^s$, with each $s \in S - J$, is a unit in A. Now $J = \bigcap J_i$, where each J_i is a prime ideal of S [8, Theorem 1.1]. Let ψ_i: $A \to R[S]/R[J_i] = R[S-J_i]$ be the natural surjection. For a fixed i, $\psi_i(u)$ is a unit in $R[S-J_i]$; thus $\psi_i(u) = r_s X^s$ for some $r_s \in U(R)$ and $s \in H$ since $R[S-J_i]$ has only trivial units. But then $\psi_i(u) = r_s X^s$ for all i since H is the group of invertible elements for each $S - J_i$. Hence $u = r_s X^s$. ∎

We recall that a torsionless grading monoid S is *seminormal* if whenever $2s$, $3s \in S$ for some $s \in \langle S \rangle$, then $s \in S$ (or equivalently, if $2s = 3t$ for some s, $t \in S$, then $s = 3u$ and $t = 2u$ for some $u \in S$). Then the monoid domain $R[S]$ is seminormal if and only if R and S are each seminormal [1, Corollary 6.2]. Similarly, S is *integrally closed* if $s \in S$ whenever $ns \in S$ for some $n \geqslant 1$ and $s \in \langle S \rangle$. Note that an integrally closed grading monoid is also seminormal. The monoid domain $R[S]$ is integrally closed if and only if R and S are each integrally closed [8, Corollary 12.11]. The *integral closure* of S is $\bar{S} = \{s \in \langle S \rangle | ns \in S \text{ for some } n \geqslant 1\}$, and the integral closure of $R[S]$ is $\bar{R}[\bar{S}]$, where $\bar{R}$ is the integral closure of R. Our next lemma shows that seminormality is preserved by certain homomorphic images.

LEMMA 2. *Let J be a prime ideal of a torsionless grading monoid S. Then $S - J$ is seminormal (resp., integrally closed) if S is seminormal (resp.,* *integrally closed*). *In particular*, $R[S]/R[J]$ *is seminormal* (*resp*., *integrally closed*) *if* $R[S]$ *is seminormal* (*resp*., *integrally closed*).

Proof. Suppose that $2s$, $3s \in S-J$ for some $s \in \langle S-J \rangle$. Then $s \in S$ since S is seminormal and thus $s \in S-J$ since J is an ideal of S. The integrally closed case may be proved similarly. The last statement of the lemma follows because $R[S]/R[J]=R[S-J]$. ∎

We next show that each seminormal monoid is the directed union of finitely generated seminormal submonoids. This is the semigroup analogue of a similar fact for seminormal integral domains (cf. the proof of [9, Theorem 1.6]; in fact, the proofs are really the same).

LEMMA 3. *Let S be a seminormal torsionless grading monoid. Then S is the directed union of finitely generated seminormal submonoids of S, each with finitely generated integral closure.*

Proof. Let K be a field. Now S is the directed union of its finitely generated submonoids T_α. Let $\bar{T}_\alpha$ be the integral closure of T_α; thus $K[\bar{T}_\alpha]$ is the integral closure of $K[T_\alpha]$. Since each $K[\bar{T}_\alpha]$ is a finitely generated $K[T_\alpha]$-module [10, Proposition 31.B], each $\bar{T}_\alpha$ is finitely generated [8, Theorem 7.7]. Let $S_\alpha=\bar{T}_\alpha \cap S$. Then $T_\alpha \subset S_\alpha \subset \bar{T}_\alpha$ and S_α is a seminormal submonoid of S. Again S_α is finitely generated since $K[S_\alpha]$ is also affine. Thus S is the directed union of the S_α's and the integral closure of each S_α is $\bar{T}_\alpha$, which is finitely generated. ∎

The following important fact about seminormal rings will be very useful (cf. [9, Theorem 1.1]).

LEMMA 4. *Let R be a seminormal integral domain with overring T and let I be the conductor ideal of R in T. Then I is a radical ideal of both R and T.*

Proof. It is sufficient to show that $x^2 \in I$ for $x \in T$ implies $x \in I$. Let $t \in T$. Then $(xt)^2$, $(xt)^3 \in I \subset R$, so $xt \in R$ since R is seminormal. Thus $xT \subset R$, so $x \in I$. ∎

Our next lemma is an elementary result about the Picard group of a group ring.

LEMMA 5. *Let R be an integral domain and let H and G be torsion-free abelian groups with* $\operatorname{rank} H \leqslant \operatorname{rank} G$. *If* $\operatorname{Pic}(R)=\operatorname{Pic}(R[G])$, *then also* $\operatorname{Pic}(R)=\operatorname{Pic}(R[H])$.

Proof. We first assume that H is a subgroup of G. Thus $R[H]$ is a subring of $R[G]$ and this inclusion induces an epimorphism α: $\operatorname{Pic}(R[H]) \to$

$\operatorname{Pic}(R[G])$ since $\operatorname{Pic}(R)=\operatorname{Pic}(R[G])$. We show that α is also injective. Let K be the quotient field of R. Suppose that I is an integral invertible ideal of $R[H]$ such that $IR[G]=fR[G]$ for some $f\in R[G]$. Since $K[H]$ is a GCD-domain [8, Theorem 14.2], $\operatorname{Pic}(K[H])=\langle 0\rangle$. Thus $IK[H]=gK[H]$ for some $g\in I$. Hence $IK[G]=gK[G]=fK[G]$; so $g=(a/b)X^sf$ for some nonzero $a,b\in R$ and $s\in G$. Then $aIR[G]=afR[G]=aX^sfR[G]=bgR[G]$, so $(aI)R[G]=(bgR[H])R[G]$. Thus $aI=bgR[H]$ by [10, 4.C(ii)] since $R[G]$ is faithfully flat (in fact, free) over $R[H]$ [8, Theorem 12.1]. Hence I is principal and α is thus injective.

Thus $\operatorname{Pic}(R)=\operatorname{Pic}(R[F])$ for any finitely generated (free) subgroup F of G. Hence $\operatorname{Pic}(R)=\operatorname{Pic}(R[F])$ for any finitely generated subgroup F of H. Since H is the directed union of such subgroups, also $\operatorname{Pic}(R)=\operatorname{Pic}(R[H])$. ∎

An integral domain R is *quasinormal* if $\operatorname{Pic}(R)=\operatorname{Pic}(R[X,X^{-1}])$ $(=\operatorname{Pic}(R[\mathbb{Z}]))$. An integrally closed domain is quasinormal (cf. [1, Corollary 5.6]) and a quasinormal integral domain is seminormal [6, Corollary 6.4], but a seminormal integral domain need not be quasinormal [13, p. 96]. Unlike the seminormal case, there does not yet seem to be any nice ring-theoretic characterization of when an integral domain is quasinormal (however, for some special cases see [6], [11], [12], or [15]). Also unlike the seminormal case, it does not seem to be known if $\operatorname{Pic}(R)=\operatorname{Pic}(R[\mathbb{Z}^n])$ for all $n\geqslant 1$ when R is quasinormal. (Or equivalently, does R quasinormal imply that $R[X,X^{-1}]$ is also quasinormal?) This remark motivates our next definition: an integral domain R is *strongly quasinormal* if $\operatorname{Pic}(R)=\operatorname{Pic}(R[\mathbb{Z}^n])$ for all $n\geqslant 1$. Thus by Lemma 5, R is strongly quasinormal if and only if $\operatorname{Pic}(R)=\operatorname{Pic}(R[G])$ for all torsion-free abelian groups G. Note that an integrally closed domain is strongly quasinormal since each $R[\mathbb{Z}^n]$ is integrally closed.

Our final lemma is a special case of our main theorem; this case has an easy proof.

LEMMA 6. *Let R be a seminormal integral domain and S a finitely generated integrally closed torsionless grading monoid with H its group of invertible elements. If* $\operatorname{Pic}(R)=\operatorname{Pic}(R[H])$, *then* $\operatorname{Pic}(R)=\operatorname{Pic}(R[S])$.

Proof. Since S is finitely generated and integrally closed, S has the form $H\oplus T$, where T has the form $\langle T\rangle\cap\mathbb{Z}_+^n$ for some $n\geqslant 0$ [8, Theorem 15.2]. Thus T is also integrally closed and $R[S]$ is seminormal. Now $A=R[S]=R[H][T]$ may be viewed as a subring of $R[H][\mathbb{Z}_+^n]$ generated by monomials over $R[H]$. In particular, A is a $\mathbb{Z}_+$-graded ring with $A_0=R[H]$. By [4, Theorem 1] $\operatorname{Pic}(A_0)=\operatorname{Pic}(A)$, hence $\operatorname{Pic}(R)=\operatorname{Pic}(A_0)=\operatorname{Pic}(A)$. ∎

2. Main Theorem

In this section we prove our main theorem which characterizes when $\mathrm{Pic}(R) = \mathrm{Pic}(R[S])$ for a monoid domain $R[S]$. Since R is a retract of $R[S]$ for any torsionless grading monoid S, $\mathrm{Pic}(R[S]) = \mathrm{Pic}(R) \oplus N\,\mathrm{Pic}(R[S])$, where $N\,\mathrm{Pic}(R[S]) = \ker(\mathrm{Pic}(R[S]) \to \mathrm{Pic}(R))$. Thus we determine necessary and sufficient conditions for $N\,\mathrm{Pic}(R[S]) = \langle 0 \rangle$. In essence, our theorem reduces the problem of deciding when $\mathrm{Pic}(R) = \mathrm{Pic}(R[S])$ for a torsionless grading monoid S to that of deciding when $\mathrm{Pic}(R) = \mathrm{Pic}(R[G])$ for G a torsion-free abelian group.

THEOREM. *Let R be an integral domain and S a nonzero torsionless grading monoid with H its group of invertible elements. Then* $\mathrm{Pic}(R) = \mathrm{Pic}(R[S])$ *if and only if $R[S]$ is seminormal and* $\mathrm{Pic}(R) = \mathrm{Pic}(R[H])$.

Proof. First suppose that $\mathrm{Pic}(R) = \mathrm{Pic}(R[S])$. Then $R[S]$ is seminormal by [1, Theorem 4.2]. We may assume that $H \neq S$ and thus $P = S - H$ is a prime ideal of S. By [8, Theorem 12.1], $R[H] = R[S-P]$ is a retract of $R[S]$ and thus the induced map $\mathrm{Pic}(R[H]) \to \mathrm{Pic}(R[S])$ is injective. Hence $\mathrm{Pic}(R) = \mathrm{Pic}(R[H])$ since $\mathrm{Pic}(R) = \mathrm{Pic}(R[S])$.

Conversely, suppose that $R[S]$ is seminormal and $\mathrm{Pic}(R) = \mathrm{Pic}(R[H])$. Thus R and S are each seminormal. We first reduce to the case in which S is finitely generated. By Lemma 3, S is the directed union of finitely generated seminormal submonoids S_α, each with finitely generated integral closure. It is thus sufficient to show that $\mathrm{Pic}(R) = \mathrm{Pic}(R[S_\alpha])$ for each α. Let H_α be the group of invertible elements of S_α. Then H_α is a subgroup of H, so $\mathrm{Pic}(R) = \mathrm{Pic}(R[H_\alpha])$ by Lemma 5. Thus the hypotheses on S carry over to each S_α, so we may assume that S is finitely generated. Our proof will be by induction on $n = \mathrm{rank}\langle S \rangle$ $(= \dim K[S]$, where K is the quotient field of R [8, Theorems 17.1 and 24.1]). The cases $n = 0$ and $n = 1$ are clear since if $n = 1$, then S is isomorphic to either $\mathbb{Z}_+$ or $\mathbb{Z}$ (cf. [8, Theorem 2.6]).

Let T be the integral closure of S. By Lemma 6, we may assume $S \neq T$. Then $B = R[T]$ is an integral overring of $A = R[S]$. Since T is finitely generated, the conductor ideal I of A in B is a nonzero S-homogeneous ideal. Moreover, $I = R[J]$, where $J = \{t \in T \mid t + T \subset S\}$ is the conductor ideal of S in T. By Lemma 4, I is a radical ideal in both A and B and hence J is a radical ideal in both S and T. We then have the following cartesian square of rings:

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ A/I & \longrightarrow & B/I \end{array}$$

By applying the Mayer–Vietoris sequence for (U, Pic) and noting that $U(B) = U(B/I)$ by Lemma 1, we obtain the following exact sequence [5, p. 482]:

$$0 \to \mathrm{Pic}(A) \to \mathrm{Pic}(B) \oplus \mathrm{Pic}(A/I) \to \mathrm{Pic}(B/I).$$

Since R is a retract of each of these rings, to show that $\mathrm{Pic}(R) = \mathrm{Pic}(A)$ it is sufficient to show that $\mathrm{Pic}(R) = \mathrm{Pic}(B)$ and $\mathrm{Pic}(R) = \mathrm{Pic}(A/I)$. Let G be the group of invertible elements of T. Then H is a subgroup of G and G/H is a torsion group, so rank $\mathrm{H} = \mathrm{rank}\ G$. Thus $\mathrm{Pic}(R) = \mathrm{Pic}(R[G])$ by Lemma 5 and hence $\mathrm{Pic}(R) = \mathrm{Pic}(R[T]) = \mathrm{Pic}(B)$ by Lemma 6. If I is actually a prime ideal of A, then $A/I = R[S-J]$ is seminormal by Lemma 2 and H is also the group of invertible elements of $S-J$. Since J is nonempty, $\dim(K[S-J] = K[S]/K[J]) < \dim K[S] = n$. Thus $\mathrm{Pic}(R) = \mathrm{Pic}(A/I)$ by the induction hypothesis, and hence also $\mathrm{Pic}(R) = \mathrm{Pic}(A)$.

Next, suppose that I is just a radical ideal of A. Then $I = P_1 \cap \cdots \cap P_r$, where each P_i is a S-homogeneous prime ideal of A. Moreover, each $P_i = R[J_i]$, where each J_i is a prime ideal of S and $J = J_1 \cap \cdots \cap J_r$ (cf. [8, Theorem 1.1], r is finite because S is finitely generated and thus $K[S]$ is noetherian). The proof that $\mathrm{Pic}(R) = \mathrm{Pic}(A/I)$ will be by induction on r; the case $r = 1$ is when I is prime and was done above. We then have the following cartesian square of rings with $L = P_1 \cap \cdots \cap P_{r-1}$:

$$\begin{array}{ccc} C_1 = A/I & \longrightarrow & A/P_r = C_2 \\ \downarrow & & \downarrow \\ C_3 = A/L & \longrightarrow & A/L + P_r = C_4. \end{array}$$

Since $C_2 = R[S - J_r]$ and $C_4 = A/R[M]$, where $M = (J_1 \cap \cdots \cap J_{r-1}) \cup J_r$ is a radical ideal of S, Lemma 1 gives $U(C_2) = U(C_4)$. The Mayer–Vietoris exact sequence for (U, Pic) then yields the following exact sequence:

$$0 \to \mathrm{Pic}(C_1) \to \mathrm{Pic}(C_2) \oplus \mathrm{Pic}(C_3) \to \mathrm{Pic}(C_4).$$

Since R is a retract of each C_i, $\mathrm{Pic}(R) = \mathrm{Pic}(C_1)$ if $\mathrm{Pic}(R) = \mathrm{Pic}(C_2)$ and $\mathrm{Pic}(R) = \mathrm{Pic}(C_3)$. Both of these equalities hold by induction hypothesis. Hence $\mathrm{Pic}(R) = \mathrm{Pic}(A/I)$, and thus $\mathrm{Pic}(R) = \mathrm{Pic}(A)$. ∎

We next give several corollaries. Recall that R is strongly quasinormal if $\mathrm{Pic}(R) = \mathrm{Pic}(R[\mathbb{Z}^n])$ for all $n \geqslant 1$. In particular, an integrally closed domain is strongly quasinormal. If each quasinormal integral domain is actually strongly quasinormal, then our theorem simplifies to $\mathrm{Pic}(R) = \mathrm{Pic}(R[S])$ if and only if S is seminormal and R is either seminormal or quasinormal, depending on whether H is, respectively, zero or nonzero.

COROLLARY 1. *Let R be a strongly quasinormal integral domain and S a torsionless grading monoid. Then the following statements are equivalent.*

(1) $\mathrm{Pic}(R) = \mathrm{Pic}(R[S])$.

(2) *$R[S]$ is seminormal.*

(3) *S is seminormal.*

COROLLARY 2. *Let K be a field and S a torsionless grading monoid. Then the following statements are equivalent.*

(1) $\mathrm{Pic}(K[S]) = \langle 0 \rangle$.

(2) *$K[S]$ is seminormal.*

(3) *S is seminormal.*

When $\langle S \rangle = \bigoplus \mathbb{Z} e_\alpha$ is a free abelian group, the monoid ring $R[S]$ may be considered as a subring of $R[\{X_\alpha, X_\alpha^{-1}\}]$ generated by monomials over R. In this context, a special interesting case of Corollary 2 is

COROLLARY 2'. *Let K be a field and R a subring of $K[X_1, X_1^{-1}, ..., X_n, X_n^{-1}]$ generated by monomials over K. Then $\mathrm{Pic}(R) = \langle 0 \rangle$ if and only if R is seminormal.*

If R is actually a subring of $K[X_1, ..., X_n]$ generated by monomials over K, then Corollary 2' is a special case of [3, Corollary 4].

COROLLARY 3. *Let R be an integral domain. Then $\mathrm{Pic}(R) = \mathrm{Pic}(R[S])$ for all seminormal torsionless grading monoids S if and only if R is strongly quasinormal.*

COROLLARY 4. *Let R be an integral domain and S a nonzero torsionless grading monoid with group of invertible elements $H = \langle 0 \rangle$. Then the following statements are equivalent.*

(1) $\mathrm{Pic}(R) = \mathrm{Pic}(R[S])$.

(2) *$R[S]$ is seminormal.*

(3) *R and S are each seminormal.*

If $H = \langle 0 \rangle$, our theorem has a much easier proof because then each T_α $(= \bar{S}_\alpha)$, and hence each S_α, is isomorphic to a submonoid of $\mathbb{Z}_+^n$, as in the proof of Lemma 6. Then each $\mathrm{Pic}(R) = \mathrm{Pic}(R[S_\alpha])$ by [4, Theorem 1] and hence $\mathrm{Pic}(R) = \mathrm{Pic}(R[S])$. If H is a direct summand of S, say $S = H \oplus T$, then T is seminormal, T has no nonzero invertible elements, and $R[S] = R[H][T]$. Thus $\mathrm{Pic}(R[S]) = \mathrm{Pic}(R[H])$ by our earlier remarks, and hence $\mathrm{Pic}(R) = \mathrm{Pic}(R[S])$ since $\mathrm{Pic}(R) = \mathrm{Pic}(R[H])$ by hypothesis. Our next example is a seminormal subring $K[S]$ of $K[\mathbb{Z}^2]$ in which the group of invertible elements H of S is not a direct summand of S. Thus the decomposition $S = H \oplus T$ for an integrally closed S need not hold for S seminormal.

EXAMPLE. Let K be a field and X and Y indeterminates over K. Let $R = K[X, XY, Y^2, Y^{-2}] = K[\{XY^n | n \in \mathbb{Z}\}, Y, Y^{-2}]$. Then R is seminormal with integral closure $K[X, Y, Y^{-1}]$. Now $R = K[S]$ with $S = \{(m, n) \in \mathbb{Z}^2 | m = 0$ and $n \in 2\mathbb{Z}$, or $m \geqslant 1\}$. S is seminormal, not integrally closed, and has group of invertible elements $H = 0 \times 2\mathbb{Z}$. We cannot decompose S as $H \oplus T$. For $T \approx S/H$ is not torsionless; for example, $2((1, 1) + H) = 2((1, 0) + H)$ while $(1, 1) + H \neq (1, 0) + H$.

Our theorem characterizes when the rank-one finitely generated projective $R[S]$-modules are extended from R. It is natural to ask when projective $R[S]$-modules of higher rank are also extended from R. The exact analogue of our theorem is not true, since there is a two-dimensional local normal domain R with $K_0(R) \neq K_0(R[X])$ [18, Section 2]. However, we will raise the following question, which generalizes a conjecture in [2, p. 11].

QUESTION. Let K be a field and S a torsionless grading monoid. Are all finitely generated projective $K[S]$-modules free if (and only if) S is seminormal?

Several special cases of our question are known to be true. For example, if $S = \mathbb{Z}_+^n$, the question has a positive answer by Quillen's solution [14] to Serre's Problem, while the case $S = \mathbb{Z}^n$ has been proved by Swan [17]. Both of those cases were generalized by Chouinard [7, Theorem 4.4] to the case in which $K[S]$ is a Krull domain with torsion divisor class group. Several more positive examples of our question are given in [2], and the techniques used in our present paper may be used to generate many more.

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